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Dual Combination Combination Anti-synchronization Of Non-identical Fractional Order Chaotic System With Different Dimension Using Scaling Matrix

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Abstract
In this manuscript Dual Combination Combination Anti-Synchronization(DCCAS) has been performed among eight non identical fractional order chaotic systems with different dimensions using scaling matrix .Based on the Lyapunov Stability Theory and Active control technique, DCCAS has been achieved.To verify our results numerical simulations have been provided, which shows that our theoretical results are in complete agreement with the graphical one.

Keywords
Combination-Combination Synchronization ; Dual Synchronization ; Anti Synchronization;Active Control;Fractional order chaotic systems

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Contents

1. Introduction

Chaos Theory[2] [7] [15] is a branch of mathematics which deals with the behavior analysis of non linear dynamical systems.An interesting property of chaotic systems,termed as 'Butterfly Effect' is sensitive dependence on initial conditions.In chaos synchronization, trajectories of two chaotic systems are made to follow the same dynamics.Synchronization was first introduced by Pecora & Caroll[9] in 1990.Recently chaos control and chaos synchronization [12][4][8][6] of fractional order dynamical systems has become an active field of study.Many techniques have been developed for control and synchronization of chaos.

From last two decades,fractional calculus[5][10] has played a major role in the study of non linear dynamical systems. The edge of fractional calculus over integer calculus is its ability to describe real systems in the interdisciplinary fields more elegantly .It also has a long range memory behavior as compared to its integer counterparts.Fractional calculus has proved useful in the fields of engineering sciences such as secure communication [14],data encryption,visco-elasticity,electro-magnetic waves,dielectric polarization etc.

The rest of the article is organized as:
Sec 2 consists of problem formulation,**Sec 3** describes the systems on which numerical simulations have been performed.**Sec 4** contains numerical simulations and discussions.**Sec 5** concludes the article.
[1].

2. Problem Formulation

We here design the dual combination combination anti-synchronization scheme.We first consider two master systems as:

$$D^{\alpha}X_1 = A_1X_1 + F_1(X_1) \tag{1}$$
$$D^{\alpha}Y_1 = B_1Y_1 + G_1(Y_1) \tag{2}$$

where $X_1 = [x_{11}, x_{12}, ...x_{1n}]^T$ and $Y_1 = [y_{11}, y_{12}, ...y_{1n}]^T$ are state vectors of master systems (1) and (2) respectively, $A_1 \in$

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$R^{n\times n}$, $B_1 \in R^{n\times n}$ are linear parts of respective systems (1) and (2),$F_1, G_1 : R^n \to R^n$ are continuous vector functions of systems (1) and (2) respectively.

Next we consider two more master systems as:

$$D^{\alpha}X_2 = A_2X_2 + F_2(X_2) \quad (3)$$
$$D^{\alpha}Y_2 = B_2X_2 + G_2(X_2) \quad (4)$$

where $X_2 = [x_{21}, x_{22}, ...x_{2n}]^T$ and $Y_2 = [y_{21}, y_{22}, ...y_{2n}]^T$ are state vectors of master systems (3) and (4) respectively, $A_2 \in R^{n\times n}$, $B_2 \in R^{n\times n}$ are linear parts of respective systems (3) and (4),$F_2, G_2 : R^n \to R^n$ are continuous vector functions of systems (3) and (4) respectively.

For master systems (1) and (2),we consider two slave systems as:

$$D^{\alpha}X_3 = A_3X_3 + F_3(X_3) + U_1 \quad (5)$$
$$D^{\alpha}Y_3 = B_3Y_3 + G_3(Y_2) + V_1 \quad (6)$$

where $X_3 = [x_{31}, x_{32}, ...x_{3n}]^T$ and $Y_3 = [y_{31}, y_{32}, ...y_{3n}]^T$ are state vectors of slave systems (5) and (6) respectively, $A_3 \in R^{n\times n}$, $B_3 \in R^{n\times n}$ are linear parts of respective systems (5) and (6),$F_3, G_3 : R^n \to R^n$ are continuous vector functions of systems (5) and (6) respectively. U_1, V_1 are control functions which are to be designed.
For master systems (3) and (4),we consider two slave systems as:

$$D^{\alpha}X_4 = A_4X_4 + F_4(X_4) + U_2 \quad (7)$$
$$D^{\alpha}Y_4 = B_4Y_4 + G_4(Y_4) + V_2 \quad (8)$$

where $X_4 = [x_{41}, x_{42}, ...x_{4n}]^T$ and $Y_4 = [y_{41}, y_{42}, ...y_{4n}]^T$ are state vectors of slave systems (7) and (8) respectively, $A_4 \in R^{n\times n}$, $B_4 \in R^{n\times n}$ are linear parts of respective systems (7) and (8),$F_4, G_4 : R^n \to R^n$ are continuous vector functions of systems (7) and (8) respectively. U_2, V_2 are control functions which are to be designed.
We define the error state function as:

$$E = (W + Z) + Q(Y + X) \quad (9)$$

where $E = [e_1, e_2]^T, W = [X_3, Y_3]^T, Z = [X_4, Y_4]^T, Y = [X_1, Y_1]^T$, $X = [X_2, Y_2]^T$

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

Therefore equation (9) simplifies to:

$$e_1 = (X_3 + X_4) + Q_1(X_2 + X_1) \quad (10)$$
$$e_2 = (Y_3 + Y_4) + Q_2(Y_2 + Y_1) \quad (11)$$

Differentiating (10) and (11) we obtain the error system as:

$$D^{\alpha}e_1 = (D^{\alpha}X_3 + D^{\alpha}X_4) + Q_1D^{\alpha}(X_2 + X_1) \quad (12)$$
$$D^{\alpha}e_2 = (D^{\alpha}Y_3 + D^{\alpha}Y_4) + Q_2D^{\alpha}(Y_2 + Y_1) \quad (13)$$

Simplifying (12) and (13) we get

$$\begin{aligned} D^{\alpha}e_1 = \; & A_3e_1 - A_3X_4 - A_3Q_1(X_2 + X_1) + F_3(X_3) + A_4e_2 \\ & - A_4X_3 - A_4Q_1(X_1 + X_2) + F_4(X_4) - Q_1[A_2X_2 \\ & + F_2(X_2) + A_1X_1 + F_1(X_1)] + U_1 + U_2 \quad (14) \\ D^{\alpha}e_2 = \; & B_3e_2 - B_3Y_4 - B_3Q_2(Y_2 + Y_1) + G_3(Y_3) + B_4e_2 \\ & - B_4Y_3 - B_4Q_2(Y_1 + Y_2) + G_4(Y_4) - Q_2[B_2Y_2 \\ & + G_2(Y_2) + B_1Y_1 + G_1(Y_1)] + V_1 + V_2 \quad (15) \end{aligned}$$

Choosing controllers as:
$U_1 = -Q_1[A_1X_1 + F_1(X_1)] + A_3X_4 + A_3Q_1(X_1 + X_2) - F_3(X_3) + K_1e_1$
$U_2 = -Q_1[A_2X_2 + F_2(X_2)] + A_4X_3 + A_4Q_1(X_1 + X_2) - F_4(X_4) + K_2e_2$
$V_1 = -Q_1[B_1Y_1 + G_1(Y_1)] + B_3Y_4 + B_3Q_1(Y_1 + Y_2) - G_3(Y_3) + K_3e_1$
$V_2 = -Q_2[B_2Y_2 + G_2(Y_2)] + B_4Y_3 + B_4Q_2(Y_1 + Y_2) - G_4(Y_4) + K_4e_2$
Then from the following lemma and theorem the desired synchronization is achieved.

Lemma: Stability Analysis Of Fractional Order Linear System
Consider the fractional order linear system as:

$$D_t^{\alpha}x(t) = Ax(t) \quad (16)$$

with initial value $x(0) = x_0 = (x_{10}, x_{20}, ..., x_{n0})$, where $x = (x_1, x_2, ..x_n)^T$ is a state vector and $\alpha \in (0,1)$ and $A \in R^{n\times n}$. Then the autonomous system (16) is said to be asymptotically stable iff $|arg(\lambda_i(A))| > \frac{\alpha\pi}{2}$,i=1,2, ...n, where $arg(\lambda_i(A))$ denotes the argument of the eigenvalues λ_i of A.
Theorem: To achieve the desired dual combination-combination anti-synchronization between chaotic systems (1)-(8) we design the control functions as:

$$\begin{aligned} U = \; & -Q_1[A_2X_2 + F_2(X_2) + A_1X_1 + F_1(X_1)] + A_3X_4 \\ & + A_4X_3 + A_3Q_1(X_2 + X_1) + A_4Q_1(X_2 + X_1) - \\ & F_3(X_3) - F_4(X_4) + K_1e_1 + K_2e_2 \quad (17) \end{aligned}$$

$$\begin{aligned} V = \; & -Q_2[B_2Y_2 + G_2(Y_2) + B_1Y_1 + G_1(Y_1)] + B_3Y_4 \\ & + B_4Y_3 + B_3Q_2(Y_2 + Y_1) + B_4Q_2(Y_2 + Y_1) - \\ & G_3(Y_3) - G_4(Y_4) + K_3e_1 + K_4e_2 \quad (18) \end{aligned}$$

where $K_1, K_2, K_3, K_4 \in R^{n\times n}$are control gain matrices.
Then the dual combination combination anti synchronization

synchronization of eight systems (1)-(8) is achieved.

Proof:

We have the error system as in (14) and (15).Substituting the designed controllers (17) and (18) in (14) and (15),the error system simplifies to:

$$D^{\alpha}e_1 = (A_3 + K_1 + A_4 + K_2)e_1 \tag{19}$$
$$D^{\alpha}e_2 = (B_3 + K_1 + B_4 + K_2)e_2 \tag{20}$$

Using the stability lemma for fractional order chaotic systems we have that the error system (17) and (18) asymptotically approaches zero if we have $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$,where λ_i are the eigen values of the matrix $A_3 + K_1 + A_4 + K_2$ and $B_3 + K_3 + B_4 + K_4$. i.e.$lim_{t\to\infty}\|e\| = 0$ i.e error approaches zero with time implying DCCAS has been achieved.

3. System Description

We have illustrated dual combination combination anti-synchronization,considering the following mathematical models.

Fractional Order Complex Lorenz System:

$$\begin{aligned}
\frac{d^{\alpha}x_{11}}{dt^{\alpha}} &= a_{11}(x_{13} - x_{11})\\
\frac{d^{\alpha}x_{12}}{dt^{\alpha}} &= a_{11}(x_{14} - x_{12})\\
\frac{d^{\alpha}x_{13}}{dt^{\alpha}} &= a_{12}x_{11} - x_{13} - x_{11}x_{15}\\
\frac{d^{\alpha}x_{14}}{dt^{\alpha}} &= a_{12}x_{12} - x_{14} - x_{12}x_{15}\\
\frac{d^{\alpha}x_{15}}{dt^{\alpha}} &= x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15}
\end{aligned} \tag{21}$$

where a_{11},a_{12},a_{13} are parameters of system.For parameter values $a_{11} = 10$,$a_{12} = 180$,$a_{13} = 1$ and initial values $(x_{11}(0), x_{12}(0), x_{13}(0), x_{14}(0), x_{15}(0)) = (2,3,5,6,9)$ and $\alpha = 0.95$ the system is chaotic as shown in Fig.1.

Fractional Order Complex T System:

$$\begin{aligned}
\frac{d^{\alpha}y_{11}}{dt^{\alpha}} &= b_{11}(y_{13} - y_{11})\\
\frac{d^{\alpha}y_{12}}{dt^{\alpha}} &= b_{11}(y_{14} - y_{12})\\
\frac{d^{\alpha}y_{13}}{dt^{\alpha}} &= (b_{12} - b_{11})y_{11} - b_{11}y_{11}y_{15}\\
\frac{d^{\alpha}y_{14}}{dt^{\alpha}} &= (b_{12} - b_{11})y_{12} - b_{11}y_{12}y_{15}\\
\frac{d^{\alpha}y_{15}}{dt^{\alpha}} &= y_{11}y_{13} + y_{12}y_{14} - b_{13}y_{15}
\end{aligned} \tag{22}$$

where b_{11},b_{12},b_{13} are parameters of system.For parameter values $b_{11} = 2.1$,$b_{12} = 30$,$b_{13} = 0.6$ and initial values $(y_{11}(0), y_{12}(0), y_{13}(0), y_{14}(0), y_{15}(0)) = (8,7,6,8,7)$ and $\alpha = 0.94$ the system is chaotic as shown in Fig.1.

Fractional Order Complex Lu System:

$$\begin{aligned}
\frac{d^{\alpha}x_{21}}{dt^{\alpha}} &= a_{21}(x_{23} - x_{21})\\
\frac{d^{\alpha}x_{22}}{dt^{\alpha}} &= a_{21}(x_{24} - x_{22})\\
\frac{d^{\alpha}x_{23}}{dt^{\alpha}} &= -x_{21}x_{25} + a_{22}x_{23}\\
\frac{d^{\alpha}x_{24}}{dt^{\alpha}} &= -x_{22}x_{25} + a_{22}x_{24}\\
\frac{d^{\alpha}x_{25}}{dt^{\alpha}} &= x_{21}x_{23} + x_{22}x_{24} - a_{23}x_{25}
\end{aligned} \tag{23}$$

where a_{21},a_{22},a_{23} are parameters of system.For parameter values $a_{21} = 40$,$a_{22} = 22$,$a_{23} = 5$ and initial values $(x_{21}(0), x_{22}(0), x_{23}(0), x_{24}(0), x_{25}(0)) = (1,2,3,4,5)$ and $\alpha = 0.95$ the system is chaotic as shown in Fig.1.

Fractional Order Complex Chen System:

$$\begin{aligned}
\frac{d^{\alpha}y_{21}}{dt^{\alpha}} &= b_{21}(y_{23} - y_{21})\\
\frac{d^{\alpha}y_{22}}{dt^{\alpha}} &= b_{21}(y_{24} - y_{22})\\
\frac{d^{\alpha}y_{23}}{dt^{\alpha}} &= (b_{23} - b_{21})y_{21} - y_{21}y_{25} + b_{23}y_{23}\\
\frac{d^{\alpha}y_{24}}{dt^{\alpha}} &= (b_{23} - b_{21})y_{22} - y_{22}y_{25} + b_{23}y_{24}\\
\frac{d^{\alpha}y_{25}}{dt^{\alpha}} &= y_{21}y_{23} + y_{22}y_{24} - b_{22}y_{25}
\end{aligned} \tag{24}$$

where b_{21},b_{22},b_{23} are parameters of system .For parameter values $b_{21} = 35$,$b_{22} = 3$,$b_{23} = 28$ and initial values $(y_{21}(0), y_{22}(0), y_{23}(0), y_{24}(0), y_{25}(0)) = (1,2,3,4,5)$ and $\alpha = 0.95$ the system is chaotic as shown in Fig.1.

Fractional Order Hyper-chaotic Xling System:

$$\begin{aligned}
\frac{d^{\alpha}x_{31}}{dt^{\alpha}} &= a_{31}(x_{32} - x_{31}) + x_{34}\\
\frac{d^{\alpha}x_{32}}{dt^{\alpha}} &= a_{32}x_{31} + x_{31}x_{13} - x_{34}\\
\frac{d^{\alpha}x_{33}}{dt^{\alpha}} &= -a_{33}x_{33} - a_{34}x_{31}x_{31}\\
\frac{d^{\alpha}x_{34}}{dt^{\alpha}} &= a_{33}x_{31}
\end{aligned} \tag{25}$$

where a_{31},a_{32},a_{33},a_{34} are parameters of system .For parameter values $a_{31} = 10$,$a_{32} = 40$,$a_{33} = 2.5$,$a_{34} = 4$ and initial values $(x_{31}(0), x_{32}(0), x_{33}(0), x_{34}(0)) = (10,40,2.5,4)$ and $\alpha = 0.95$ the system is hyper-chaotic as shown in Fig.1.

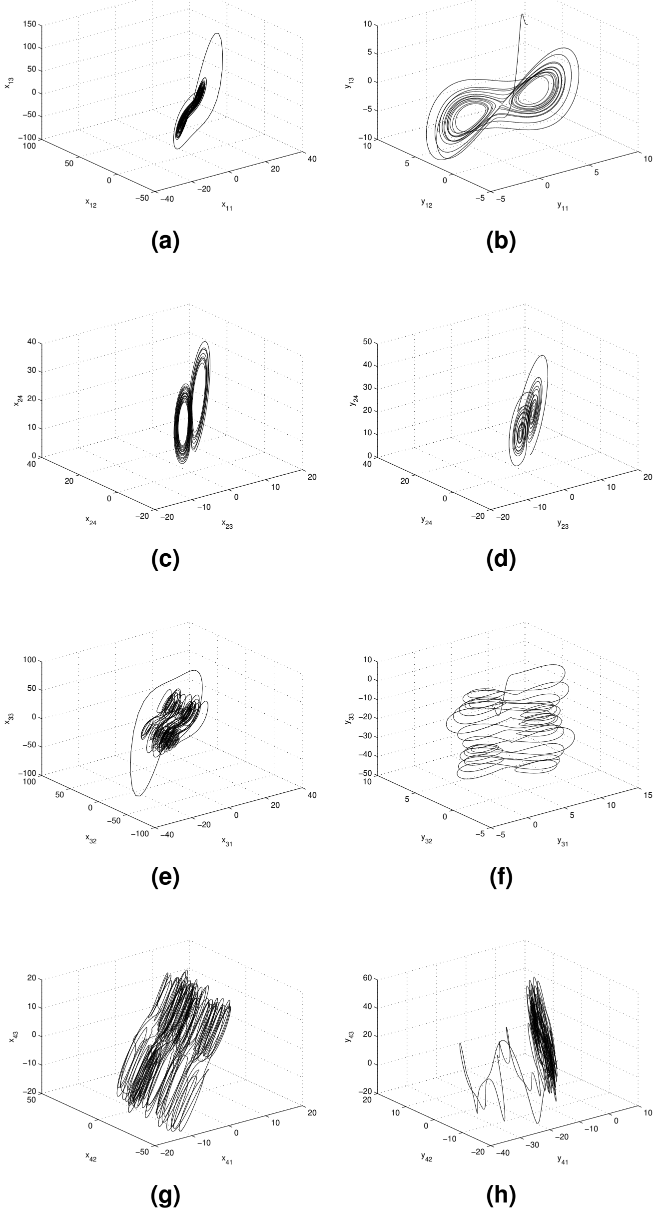

Fig 1: Phase portraits of master and slave systems (a)Complex Lorenz chaotic system in $x_{11}-x_{12}-x_{13}$ planes (b)Complex T chaotic system in $x_{21}-x_{22}-x_{23}$ planes (c)Complex Lu chaotic system in $x_{31}-x_{32}-x_{33}$ planes (d)Complex Chen chaotic system in $x_{41}-x_{42}-x_{43}$ planes (e)Xling chaotic system in $y_{11}-y_{12}-y_{13}$ planes (f)Vanderpol chaotic system in $y_{21}-y_{22}-y_{23}$ planes (g)Rabinovich chaotic system in $y_{31}-y_{32}-y_{33}$ planes (h)Rikitake chaotic system in $y_{41}-y_{42}-y_{43}$ planes

Fractional Order Vanderpol Hyperchaotic System:

$$
\begin{aligned}
\frac{d^\alpha y_{31}}{dt^\alpha} &= y_{32}\\
\frac{d^\alpha y_{32}}{dt^\alpha} &= -(b_{31}+b_{32}y_{33})y_{31}-(b_{31}+b_{32}y_{33})y_{31}^3-b_{33}y_{32}+b_{34}y_{33}\\
\frac{d^\alpha y_{33}}{dt^\alpha} &= y_{34} \qquad (26)\\
\frac{d^\alpha y_{34}}{dt^\alpha} &= -b_{35}y_{33}+b_{36}(1-y_{33}y_{33})y_{34}+b_{37}y_{31}
\end{aligned}
$$

where b_{31},b_{32},b_{33},b_{34},b_{35},b_{36},b_{37} are parameters of system.For parameter values $b_{31}=10$,$b_{32}=3$,$b_{33}=0.4$,$b_{34}=70$,$b_{35}=1$,$b_{36}=5$,$b_{37}=0.1$ and initial values $(y_{31}(0),y_{32}(0),$ $y_{33}(0),y_{34}(0))=(0.1,-0.5,0.1,-0.5)$ and $\alpha=0.95$ the system is hyper-chaotic as shown in Fig.1.

Fractional Order Rabinovich Hyperchaotic System:

$$
\begin{aligned}
\frac{d^\alpha x_{41}}{dt^\alpha} &= (-a_{41}x_{41}+a_{42}x_{42})+x_{42}x_{23}\\
\frac{d^\alpha x_{42}}{dt^\alpha} &= a_{42}x_{41}-a_{43}x_{42}-x_{41}x_{43}+x_{44}\\
\frac{d^\alpha x_{43}}{dt^\alpha} &= -a_{44}x_{43}+x_{41}x_{42} \qquad (27)\\
\frac{d^\alpha x_{44}}{dt^\alpha} &= -a_{45}x_{42}
\end{aligned}
$$

where a_{41},a_{42},a_{43},a_{44},a_{45} are parameters of system .For parameter values $a_{41}=34$,$a_{42}=6.75$,$a_{43}=1$,$a_{44}=1$,$a_{45}=2$ and initial values $(x_{41}(0),x_{42}(0),$ $x_{43}(0),x_{44}(0))=(5.5,-1.25,8.4,2.75)$ and $\alpha=0.95$ the system is hyper-chaotic as shown in Fig.1.

Fractional Order Hyperchaotic Rikitake System:

$$
\begin{aligned}
\frac{d^\alpha y_{41}}{dt^\alpha} &= -b_{41}y_{41}+y_{42}y_{43}-b_{42}y_{44}\\
\frac{d^\alpha y_{42}}{dt^\alpha} &= -b_{41}y_{42}+y_{41}(y_{43}-b_{43})-b_{42}y_{44}\\
\frac{d^\alpha y_{43}}{dt^\alpha} &= 1-y_{41}y_{42} \qquad (28)\\
\frac{d^\alpha y_{44}}{dt^\alpha} &= b_{44}y_{42}
\end{aligned}
$$

where b_{41},b_{42},b_{43},b_{44} are parameters of system .For parameter values $b_{41}=1$,$b_{42}=1.7$,$b_{43}=1$,$b_{44}=0.7$ and initial values $(y_{41}(0),y_{42}(0),$ $y_{43}(0),y_{44}(0))=(3.5,1.7,-4.5,2.8)$ and $\alpha=0.95$ the system is hyper-chaotic as shown in Fig.1.

4. Numerical Simulations & Discussions

We consider (21),(23),(25),(27) as master systems and (22),(24),(26),(28) as slave systems.From Section 2 we have for the above defined systems the following matrices:

$$A_1=\begin{bmatrix}-a_{11} & 0 & a_{11} & 0 & 0\\ 0 & -a_{11} & 0 & a_{11} & 0\\ a_{12} & 0 & -1 & 0 & 0\\ 0 & a_{12} & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -a_{13}\end{bmatrix},$$

$$A_2=\begin{bmatrix}-a_{21} & 0 & a_{21} & 0 & 0\\ 0 & -a_{21} & 0 & a_{21} & 0\\ 0 & 0 & a_{22} & 0 & 0\\ 0 & 0 & 0 & a_{22} & 0\\ 0 & 0 & 0 & 0 & -a_{23}\end{bmatrix}$$

$$A_3=\begin{bmatrix}-a_{31} & a_{31} & 0 & 1\\ 0 & a_{33} & 0 & 0\\ 0 & 0 & -a_{32} & 0\\ 0 & 0 & 0 & a_{34}\end{bmatrix} A_4=\begin{bmatrix}-a_{41} & a_{42} & 0 & 0\\ a_{42} & -a_{43} & 0 & 1\\ -a_{44} & 0 & 0 & 0\\ 0 & -a_{45} & 0 & 0\end{bmatrix}$$

$$F_1(X_1)=\begin{bmatrix}0\\ 0\\ -x_{11}x_{15}\\ -x_{12}x_{15}\\ x_{11}x_{13}+x_{12}x_{14}\end{bmatrix} F_2(X_2)=\begin{bmatrix}0\\ 0\\ -x_{21}x_{25}\\ -x_{22}x_{25}\\ x_{21}x_{23}+x_{22}x_{24}\end{bmatrix}$$

$$F_3(X_3)=\begin{bmatrix}0\\ x_{31}x_{33}\\ -a_{34}x_{31}x_{31}\\ 0\end{bmatrix} F_4(X_4)=\begin{bmatrix}x_{42}x_{43}\\ -x_{41}x_{43}\\ x_{41}x_{42}\\ 0\end{bmatrix}$$

$$B_1=\begin{bmatrix}-b_{11} & 0 & b_{11} & 0 & 0\\ 0 & -b_{11} & 0 & b_{11} & 0\\ (b_{12}-b_{11}) & 0 & 0 & 0 & 0\\ 0 & (b_{12}-b_{11}) & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -b_{13}\end{bmatrix}$$

$$B_2=\begin{bmatrix}-b_{21} & 0 & b_{21} & 0 & 0\\ 0 & -b_{21} & 0 & b_{21} & 0\\ (b_{23}-b_{21}) & 0 & b_{23} & 0 & 0\\ 0 & (b_{23}-b_{21}) & 0 & b_{23} & 0\\ 0 & 0 & 0 & 0 & -b_{22}\end{bmatrix}$$

$$B_3=\begin{bmatrix}0 & 0 & 0 & 0\\ -b_{31} & -b_{33} & b_{34} & 0\\ 0 & 0 & 0 & 1\\ b_{37} & 0 & -b_{35} & b_{36}\end{bmatrix} B_4=\begin{bmatrix}-b_{41} & 0 & 0 & -b_{42}\\ -b_{43} & -b_{41} & 0 & -b_{42}\\ 0 & 0 & 0 & 0\\ 0 & b_{44} & 0 & 0\end{bmatrix}$$

$$G_1(Y_1)=\begin{bmatrix}0\\ 0\\ -b_{11}y_{11}y_{15}\\ -b_{11}y_{12}y_{15}\\ y_{11}y_{13}+y_{12}y_{14}\end{bmatrix} G_2(Y_2)=\begin{bmatrix}0\\ 0\\ -y_{21}y_{25}\\ -y_{22}y_{25}\\ y_{21}y_{23}+y_{22}y_{24}\end{bmatrix} G_3(Y_3)=$$

$$\begin{bmatrix}0\\ -y_{31}y_{33}\\ y_{31}y_{32}\\ -y_{32}y_{33}\end{bmatrix} G_4(Y_4)=\begin{bmatrix}y_{42}y_{43}\\ y_{41}y_{43}\\ -y_{41}y_{42}\\ 0\end{bmatrix}$$

We have considered here:

$$Q_1=Q_2=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 1\end{bmatrix} \tag{29}$$

Therefore we get the error function as:

$e_{11}=(x_{31}+x_{41})+(x_{21}+x_{11})$
$e_{12}=(x_{32}+x_{42})+(x_{22}+x_{12})$
$e_{13}=(x_{33}+x_{43})+(x_{23}+x_{13})$
$e_{14}=(x_{34}+x_{44})+(x_{24}+x_{14})+(x_{25}+x_{15})$
$e_{21}=(y_{31}+y_{41})+(y_{21}+y_{11})$
$e_{22}=(y_{32}+y_{42})+(y_{22}+y_{12})$
$e_{23}=(y_{33}+y_{43})+(y_{23}+y_{13})$
$e_{24}=(y_{34}+y_{44})+(y_{24}+y_{14})+(y_{25}+y_{15})$
where $e_1=[e_{11},e_{12},e_{13},e_{14}]^T$ and $e_2=[e_{21},e_{22},e_{23},e_{24}]^T$.

Choosing the control functions $U=U_1+U_2, V=V_1+V_2$ as given in (17) & (18):

$$U_1+U_2=\begin{bmatrix}u_{11}+u_{21}\\ u_{12}+u_{22}\\ u_{13}+u_{23}\\ u_{14}+u_{24}\end{bmatrix} \tag{30}$$

$$=\begin{bmatrix}a_{11}x_{11}-a_{11}x_{13}-a_{31}x_{41}+a_{31}x_{42}+x_{44}-a_{31}\\ (x_{12}+x_{22})+a_{31}(x_{12}+x_{22})+(x_{14}+x_{24}+x_{15}+x_{25})\\ -a_{31}e_{12}-e_{14}+a_{21}x_{21}-a_{21}x_{23}-a_{41}x_{31}\\ +a_{42}x_{32}-x_{42}x_{43}-a_{41}(x_{11}+x_{21})+a_{42}(x_{12}+x_{22})-a_{42}e_{12}\\ \\ a_{11}x_{12}-a_{11}x_{14}+a_{32}x_{41}-x_{44}-x_{31}x_{33}+a_{32}\\ (x_{11}+x_{21})-(x_{14}+x_{24}+x_{15}+x_{25})-a_{32}e_{11}+a_{21}x_{22}-a_{21}\\ x_{24}+a_{42}x_{31}+a_{42}x_{32}+x_{34}+x_{41}x_{43}+a_{42}(x_{11}\\ +x_{21})-a_{43}(x_{12}+x_{22})+(x_{14}+x_{24}+x_{15}+x_{25})-a_{42}e_{11}\\ \\ -a_{12}x_{11}+x_{13}+x_{11}x_{15}-a_{33}x_{43}+a_{34}x_{31}x_{31}\\ -a_{33}(x_{13}+x_{23})-a_{22}x_{23}+x_{21}x_{25}-a_{44}x_{33}-x_{41}\\ x_{42}-a_{44}(x_{13}+x_{23})\\ \\ -a_{12}x_{12}+x_{14}+a_{13}x_{15}+x_{12}x_{15}+x_{11}x_{13}+x_{12}\\ x_{14}+a_{33}x_{41}+a_{33}e_{11}-e_{14}-a_{22}x_{24}+a_{21}x_{25}\\ +x_{22}x_{25}-x_{21}x_{23}-x_{22}x_{24}-a_{45}x_{32}-a_{45}(x_{12}\\ +x_{22})+a_{45}e_{12}\end{bmatrix}$$

$$V_1+V_2=\begin{bmatrix}v_{11}+v_{21}\\ v_{12}+v_{22}\\ v_{13}+v_{23}\\ v_{14}+v_{24}\end{bmatrix}= \tag{31}$$

$$\begin{bmatrix}b_{11}y_{11}-b_{11}y_{13}+y_{42}+(y_{12}+y_{22})-e_{22}-b_{21}y_{21}-b_{21}y_{23}-b_{41}\\ (y_{11}+y_{21})-b_{42}((y_{14}+y_{24}+y_{15}+y_{25})+b_{42}e_{24}\\ \\ b_{11}y_{12}-b_{11}y_{14}-b_{31}y_{41}-b_{33}y_{42}+b_{34}y_{43}+b_{32}y_{33}y_{31}+\\ (b_{31}+b_{32}y_{33})y_{31}y_{31}y_{31}-b_{31}(y_{11}+y_{21})-b_{33}(y_{12}+y_{22})\\ +b_{34}(y_{13}+y_{23})+b_{31}e_{21}-b_{34}e_{23}\\ \\ -(b_{12}-b_{11})y_{11}+b_{11}y_{11}y_{15}+y_{44}+(y_{14}+y_{24}+y_{15}+y_{25})-e_{23}\\ -e_{24}-(b_{23}-b_{21})y_{21}-b_{23}y_{23}+y_{21}y_{25}+y_{41}y_{42}\\ \\ -(b_{12}-b_{11})y_{12}+b_{13}y_{15}+b_{11}y_{12}y_{15}-y_{11}y_{13}-y_{12}y_{14}+b_{37}\\ y_{41}-b_{35}y_{43}+b_{36}y_{44}+b_{36}y_{33}y_{33}y_{34}+b_{37}(y_{11}+y_{21})-\\ b_{35}(y_{13}+y_{23})+b_{36}(y_{14}+y_{24}+y_{15}+y_{25})-b_{37}e_{21}+b_{35}e_{23}\\ -(b_{36}+1)e_{24}-(b_{23}-b_{21})y_{22}-b_{23}y_{24}+b_{22}y_{25}-y_{21}y_{23}-y_{22}y_{24}\\ +b_{44}y_{32}+y_{44}(y_{12}+y_{22})-b_{44}e_{22}\end{bmatrix}$$

From (19) and (20),we obtain the error system as:

$$\frac{d^\alpha e_1}{dt^\alpha} = (A_3 + K_1 + A_4 + K_2)e_1$$
$$\frac{d^\alpha e_2}{dt^\alpha} = (B_3 + K_1 + B_4 + K_2)e_2 \tag{32}$$

Choosing control gain matrices as:

$$K_1 = \begin{bmatrix} 0 & -a_{31} & 0 & -1 \\ -a_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_{33} & 0 & 0 & -1 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & -a_{42} & 0 & 0 \\ -a_{42} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{45} & 0 & 0 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ b_{31} & 0 & -b_{34} & 0 \\ 0 & 0 & -1 & -1 \\ -b_{37} & 0 & b_{35} & -b_{36}-1 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 0 & 0 & 0 & b_{42} \\ b_{43} & 0 & 0 & b_{42} \\ 0 & 0 & 0 & 0 \\ 0 & -b_{44} & 0 & 0 \end{bmatrix}$$

the systems (21)-(28) achieve the dual combination combination anti synchronization as shown in Fig. 2.Also the error plot of the system converges to zero for initial conditions of error system as $(e_{11}, e_{12}, e_{13}, e_{14}) = (9.5, 5.75, 19.4, 30.75)$ and $(e_{21}, e_{22}, e_{23}, e_{24}) = (12.6, 10.2, 4.5, 26.3)$ as shown in Fig. 3.

5. Conclusion

In this paper the dual combination combination anti-synchronization has been achieved among eight fractional order chaotic systems with different dimensions using scaling matrix.The synchronization has been achieved using the stability result of fractional order systems.
This technique would be useful in the field of secure communication because of the complexity of the systems involved.

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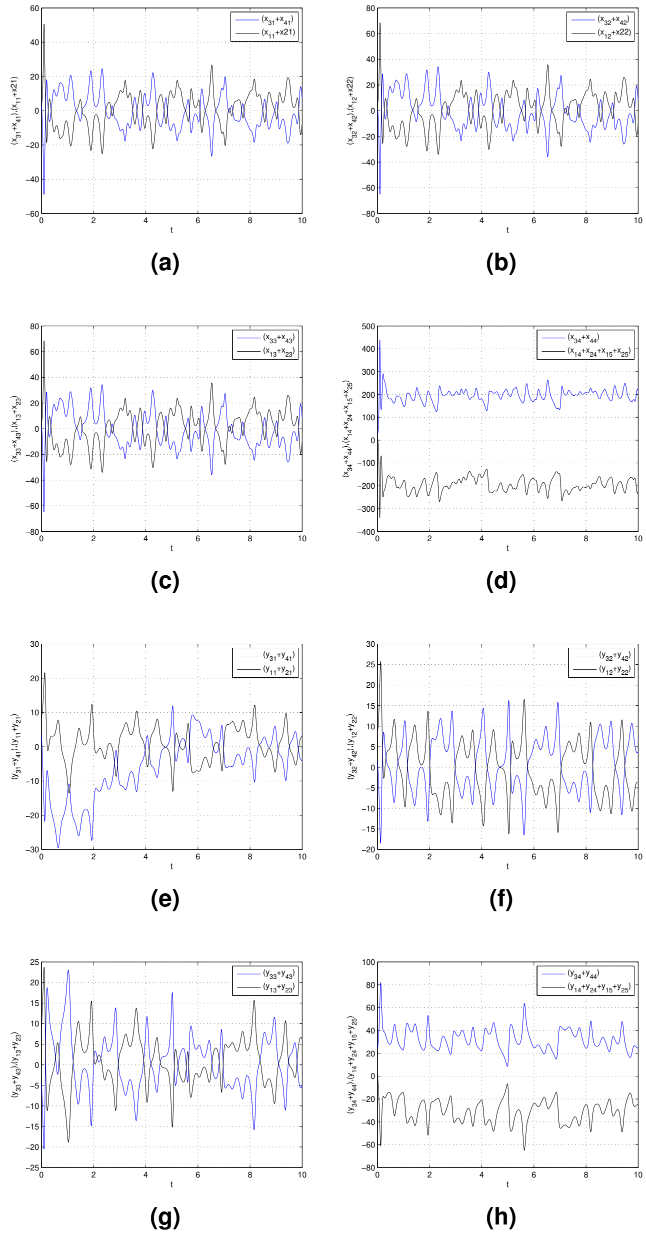
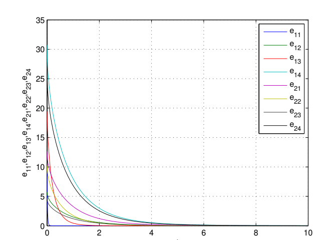


Fig 2: Dual combination combination anti-synchronized trajectories of master and slave systems



(a)

Fig 3: The error plot of the system

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